

Conservative systems

This chapter on conservative systems is to a great extent based on reference [Oh] and [Ot]. Systems that are not dissipative, no shrinkage in phase space are called **conservative systems**. They have a conserved quantity.

In a physical context described by classical mechanics the conserved quantity is energy. The system is isolated with no energy flow into and out of the system and no friction. The description of a conservative system in classical mechanics is best done in the **Hamiltonian formulation**.

Newtonian formulation for a particle with mass m , coordinates \mathbf{r} affected by a force $\mathbf{F}(\mathbf{r})$:

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$$

For a curl-free force with $\nabla \times \mathbf{F} = \mathbf{0}$ the work to move a particle around a loop is zero and it's possible to introduce a potential energy $E_p(\mathbf{r})$ such that $\mathbf{F} = -\nabla E_p$ and $m\ddot{\mathbf{r}} + \nabla E_p = 0$.

$$\mathbf{p} \equiv m\dot{\mathbf{r}} \rightarrow \begin{cases} m\dot{\mathbf{r}} = \mathbf{p} \\ \dot{\mathbf{p}} = -\nabla E_p \end{cases} \text{ with kinetic energy } E_k \equiv \frac{\mathbf{p}^2}{2m} \rightarrow \frac{\partial E_k}{\partial p_i} = \frac{p_i}{m} = \dot{r}_i$$

$$\text{Total energy : } H = E_k + E_p \rightarrow \begin{cases} \dot{r}_i = \frac{\partial H(\mathbf{r}, \mathbf{p})}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H(\mathbf{r}, \mathbf{p})}{\partial r_i} \end{cases} \begin{array}{l} \text{An ODE of first order with a unique solution} \\ \text{once initial conditions } \mathbf{r}(0), \mathbf{p}(0) \text{ are given.} \\ \text{These equations are derived under assumption} \\ \text{of a velocity-independent potential.} \end{array}$$

The Hamiltonian formulation is a generalization to generalized coordinates q_i , generalized momenta p_i and a scalar **Hamiltonian function** $H(\mathbf{q}, \mathbf{p}, t)$ that represents the energy of the system. The general coordinate q_i could be an angular variable which would make the momentum variable p_i an angular momentum.

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial q_i} \\ \frac{dq_i}{dt} = \frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial p_i} \end{cases}$$

If the Hamiltonian has no explicit time dependence $H = H(\mathbf{p}, \mathbf{q})$ then:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} = \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} + \frac{\partial H}{\partial \mathbf{p}} \cdot \left(-\frac{dH}{d\mathbf{q}}\right) = 0$$

As \mathbf{p} and \mathbf{q} vary with time $H(\mathbf{p}, \mathbf{q})$ remains constant. This conserved quantity that is used in the definition of a conservative system is the energy E of the system. $E = H(\mathbf{p}, \mathbf{q})$ is a constant of motion.

In the general case p_i and q_i are n -dimensional vectors and the Hamiltonian system is an ODE of first order in $2n$ dimensions. To express this in a unified manner we introduce a new $2n$ -dimensional vector $\tilde{\mathbf{x}} = (\mathbf{p}, \mathbf{q})$ and a vector function derived from the scalar Hamiltonian, $\mathbf{F}(\tilde{\mathbf{x}}, t) = \mathbf{S}_{2n} \cdot \partial H / \partial \tilde{\mathbf{x}}$ where \mathbf{S}_{2n} is a $2n \times 2n$ matrix composed of zero-matrices and identity matrices:

$$\mathbf{S}_{2n} = \begin{pmatrix} \mathbf{0}_n & -\mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0}_n \end{pmatrix} \quad \frac{\partial H}{\partial \tilde{\mathbf{x}}} = \begin{pmatrix} \partial H / \partial \mathbf{p} \\ \partial H / \partial \mathbf{q} \end{pmatrix} \rightarrow \text{Hamilton's equations : } \frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{S}_{2n} \cdot \frac{\partial H}{\partial \tilde{\mathbf{x}}}$$

A basic property of Hamilton's equations is that they preserve $2n$ -dimensional volumes in phase space. This follows from the divergence of $\mathbf{F}(\tilde{\mathbf{x}})$.

$$\frac{\partial}{\partial \tilde{\mathbf{x}}} \cdot \mathbf{F} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(-\frac{\partial H}{\partial \mathbf{q}} \right) + \frac{\partial}{\partial \mathbf{q}} \cdot \left(\frac{\partial H}{\partial \mathbf{p}} \right) = 0$$

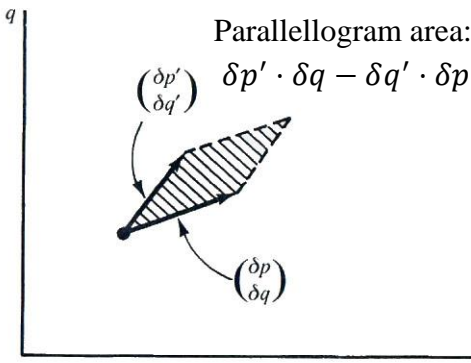
Take a closed surface $\delta S(0)$ in $(2n, 2n)$ -phase space and evolve all points according to Hamilton's equations and the volume of the endpoints $S(t)$ will not change

$$\frac{d}{dt} \int_{S_t} \tilde{\mathbf{x}} d^{2n} \tilde{\mathbf{x}} = \int_{S_t} \frac{d\tilde{\mathbf{x}}}{dt} \cdot dS = \int_{S_t} \mathbf{F} \cdot dS = \int_{S_t} \frac{\partial}{\partial \tilde{\mathbf{x}}} \cdot \mathbf{F} d^{2n} \tilde{\mathbf{x}} = 0$$

Conservation of volume \Rightarrow Hamiltonian systems do not have attractors like dissipative systems.

The incompressibility of phase space volume in Hamiltonian systems is known as **Liouville's theorem**.

The basic property that gives phase-space-preservation is the **symplectic** properties based on the matrix \mathbf{S}_{2n} .



Consider three orbits a, b, c that span an infinitesimal time-independent area spanned by vectors $(\delta p, \delta q)$ and $(\delta p', \delta q')$:

$$\frac{d}{dt} (\delta p' \cdot \delta q - \delta q' \cdot \delta p) = \frac{d}{dt} (\delta \tilde{\mathbf{x}}^T \cdot \mathbf{S}_{2n} \cdot \delta \tilde{\mathbf{x}}) = 0$$

$$\frac{d}{dt} (\delta \tilde{\mathbf{x}}^T \cdot \mathbf{S}_{2n} \cdot \delta \tilde{\mathbf{x}}) = \dots = \delta \tilde{\mathbf{x}}^T \left(\left(\frac{\partial^2 H}{\partial \tilde{\mathbf{x}} \partial \tilde{\mathbf{x}}} \right)^T \mathbf{S}_{2n}^T \mathbf{S}_{2n} + \mathbf{S}_{2n} \mathbf{S}_{2n} \frac{\partial^2 H}{\partial \tilde{\mathbf{x}} \partial \tilde{\mathbf{x}}} \right) \delta \tilde{\mathbf{x}} = 0$$

Where $\mathbf{S}_{2n} \cdot \mathbf{S}_{2n} = -\mathbf{I}_{2n}$, $\mathbf{S}_{2n}^T = -\mathbf{S}_{2n}$ and symmetry of $\partial^2 H / \partial \tilde{\mathbf{x}} \partial \tilde{\mathbf{x}}$ was used.

\therefore The infinitesimal area δA is preserved during the flow for a symplectic system.

For $n = 1$ Liouville's theorem and the symplectic condition are equivalent.

For $n > 1$ they are not but the symplectic condition implies volume conservation.

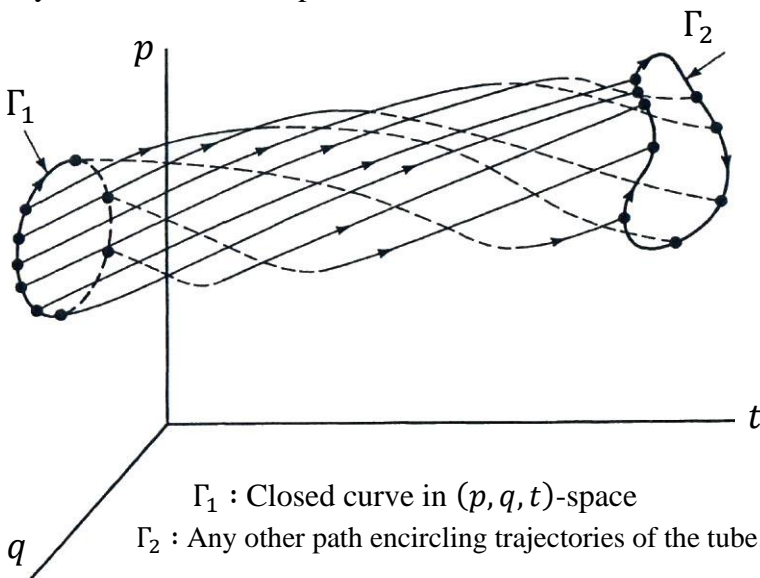
The symplectic condition is the more fundamental requirement for Hamiltonian mechanics

A general $2n \times 2n$ matrix \mathbf{A} is symplectic if $\mathbf{S}_{2n} = \mathbf{A}^T \cdot \mathbf{S}_{2n} \cdot \mathbf{A}$

Eigenvalues of a symplectic matrix are roots of $D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$, a polynomial in λ of degree $2n$.

$$\mathbf{S}_{2n} = \mathbf{A}^T \mathbf{S}_{2n} \mathbf{A} \rightarrow \mathbf{A} = \mathbf{S}_{2n}^{-1} (\mathbf{A}^T)^{-1} \mathbf{S}_{2n} \rightarrow D(\lambda) = \det(\mathbf{A}^{-1} + \lambda \mathbf{I})$$

\mathbf{A} and \mathbf{A}^{-1} have the same eigenvalues and since eigenvalues of \mathbf{A}^{-1} and \mathbf{A} are inverses of each other the eigenvalues must come in pairs (λ, λ^{-1}) and since Lyapunov exponents are obtained by logarithms ($h = \ln |\lambda|$) they must also come in pairs $\pm h$.



Poincaré-Cartans Integral theorem:

$$\oint_{\Gamma_1} (pdq - Hdt) = \oint_{\Gamma_2} (pdq - Hdt)$$

If $H(p, q)$ is independent of t :

$$\oint_{\Gamma_1} pdq = \oint_{\Gamma_2} pdq$$

Another consequence of conservation of phase space volume is **Poincaré recurrence theorem**. For a time-independent Hamiltonian $H(p, q)$ where all orbits are bounded, if for instance there are no orbits $E = H(p, q)$ with $|p| \rightarrow \infty$ or $|q| \rightarrow \infty$. Pick any point in phase space and surround it with a ball R_ϵ . If there are points which leave the initial ball there will always be some of these which will return to R_ϵ no matter how small ϵ .

The time map $\mathcal{M}_T(\tilde{\mathbf{x}}(t), t) = \tilde{\mathbf{x}}(t + T)$ has as a derivative a matrix $\partial\mathcal{M}_T/\partial\tilde{\mathbf{x}}$ that is a symplectic matrix.

Two special cases of interest are:

1. The Hamiltonian is periodic in t , $H(p, q, t) = H(p, q, t + T)$
2. The Hamiltonian has no explicit dependence on time, $H(p, q, t) = H(p, q)$

1. Make t a state variable ξ , (p, q, ξ) represents a state in $2n + 1$ dimensions with extra equation $d\xi/dt = 1$
 ξ is an angle variable replaceable in H by $\bar{\xi} = \xi \bmod T$.

Use the surface of section technique from the Poincaré map on surface $\xi = t_0 \in [0, T[$.

$\mathcal{M}_T(\tilde{\mathbf{x}}, t_0) = \mathcal{M}_T(\tilde{\mathbf{x}}, t_0 + nT) \rightarrow$ Surface of section map $\mathbf{M}(\tilde{\mathbf{x}}) = \mathcal{M}_T(\tilde{\mathbf{x}}, t_0)$ is symplectic.

Both 1 and 2 have symplectic maps $\tilde{\mathbf{x}}_{n+1} = \mathbf{M}(\tilde{\mathbf{x}}_n)$ with Lyapunov exponents $\pm h_j$.

Integrable systems

A Hamiltonian with no explicit time dependence $H(p, q)$ is symmetric under translation in t and the Hamiltonian equations imply that $dH/dt = 0$ and $E = H(p, q)$ is an invariant of the trajectory. These orbits lie on a $(2n - 1)$ -dimensional energy surface $E = H(p, q)$.

Every symmetry of $H(p, q)$ reduces the dimension one step of the manifold where an orbit can exist. These symmetries can be obvious like a translational symmetry in a Cartesian coordinate, that coordinate is absent from the Hamiltonian. This gives conserved momentum in the missing coordinate's direction. Another obvious symmetry is rotational invariance that give conserved angular momentum. Other symmetries are less obvious.

A condition that $f(p, q)$ is a constant of motion, connected to some symmetry is that the **Poisson bracket** of f and H : $[f, H]$ equals zero. The Poisson bracket of two functions g_1, g_2 depending on p and q is defined as:

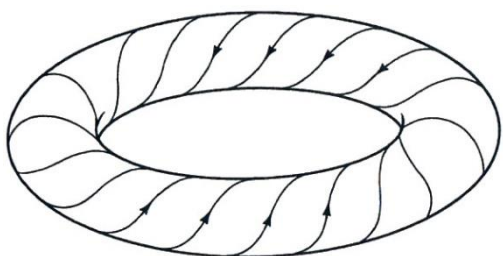
$$[g_1, g_2] \equiv \frac{\partial g_1}{\partial q} \cdot \frac{\partial g_2}{\partial p} - \frac{\partial g_1}{\partial p} \cdot \frac{\partial g_2}{\partial q}$$

A time-independent Hamiltonian is **integrable** if it has n independent global constants of motion f_1, \dots, f_n .

One of them will be the Hamiltonian itself, indexed as number one, $f_1(p, q) = H(p, q)$.

With n independent constants of motion the motion will be restricted to the n -dimensional surface $f_i(p, q) = K_i$ where $i = 1, 2, \dots, n$ with $[f_i, f_j] = 0$ when $i \neq j$.

These restrictions give a certain topology for the surface where the orbits live, an n -dimensional torus after a suitable canonical coordinate transformation. Picture shows an orbit on a 2-torus.



Phase space can then be viewed as consisting of n -tori almost all of which (in a Lebesgue measure sense) are filled with quasi-periodic orbits with n frequencies.

The set of orbits that are periodic have zero Lebesgue measure (zero phase space volume) but they are still dense in phase space, (arbitrarily near any torus with quasi-periodic orbits).

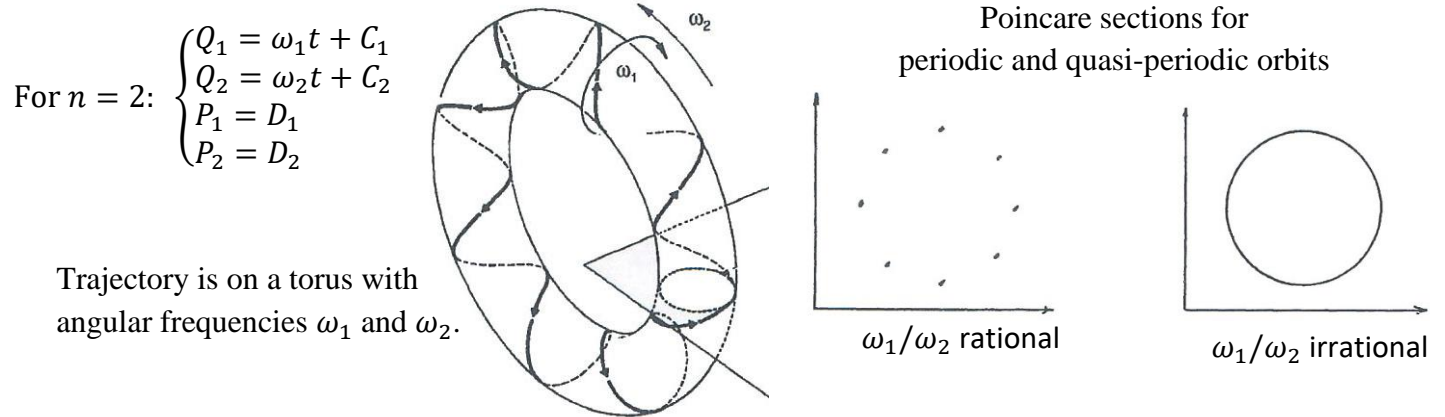
To illustrate integrable systems and canonical transformations let's look at the harmonic oscillator with a restoring force proportional to the extension from the equilibrium position:

$$m\ddot{x} = -kx \rightarrow \begin{cases} q = x \\ p = m\dot{x} \end{cases} \rightarrow H = \frac{1}{2} \left(kq^2 + \frac{p^2}{m} \right) \rightarrow \begin{cases} q = \frac{1}{m} \sqrt{2P} \sin Q \\ P = \mu \sqrt{2P} \cos Q \end{cases} \rightarrow H = \omega P \rightarrow \begin{cases} Q = \omega t + C \\ P = D \end{cases}$$

Canonical transformation with $\mu = (km)^{1/4}$

$\omega = \sqrt{k/m}$
 P is a constant of motion

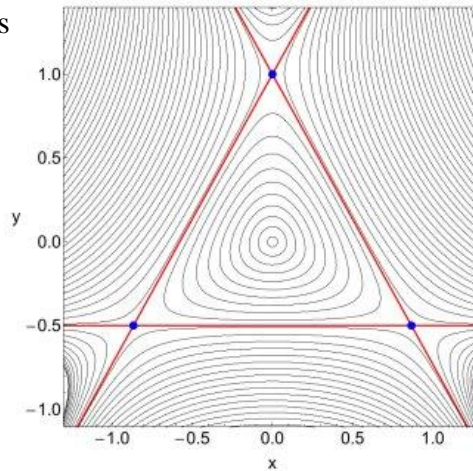
A **separable system** $H(\mathbf{q}, \mathbf{p}) = H_1(q_1, p_1) + H_2(q_2, p_2) + \dots + H_n(q_n, p_n)$ is integrable. It divides into n integrable subsystems with individual energies E_1, E_2, \dots, E_n as constants of motion.



These Poincaré sections are typical for systems with a second constant of motion. If there is no extra constant of motion there will be no extra restricting surface in phase space and the points can be all over the section with chaotic dynamics. This makes the Poincaré a valuable tool for discovering hidden constants of motion.

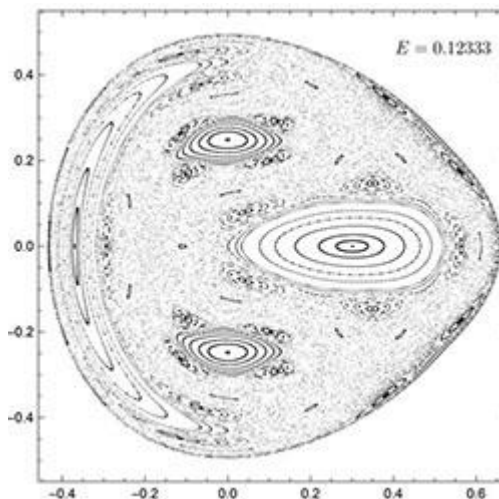
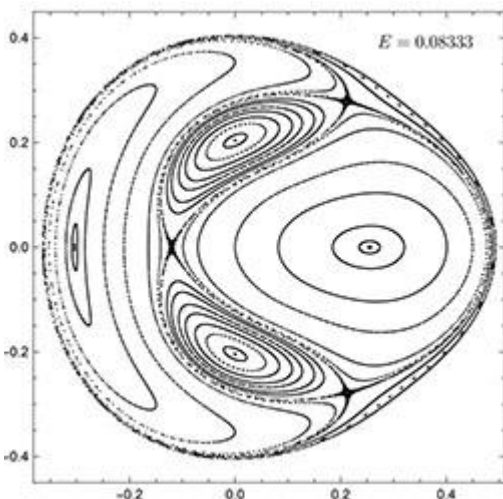
This method was used by Hénon and Heiles in 1964 with a problem from astronomy with a potential $V(x, y)$.

$$V(x, y) = \frac{1}{2}(x^2 + y^2 + 2x^2y - 2y^3/3) = \rho^2/2 + \rho^3 \sin^3 3\theta / 3$$



Equipotential curves of the Hénon-Heiles potential $V(x, y)$.

Poincaré sections of the Hénon-Heiles potential



For low energies the trajectories are restricted to closed loops while for higher energies there is a combination of orbits restricted to loops in the Poincaré section and orbits spreading out all over the cross section.

The Kolmogorov-Arnold-Moser theorem

The situation of how ordered and chaotic orbits mix in Hamiltonian systems is handled by the KAM-theorem. Kolmogorov formulated it in 1954 and Arnold and Moser proved different versions of it in 1962 and 1963.

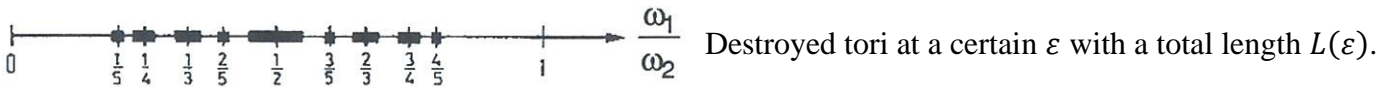
The KAM-theorem deals with integrability. One view could be that integrability is general and that all the constants of motion must exist even though they might be hard to find. Another possible view would be that if an integrable system $H_0(\mathbf{p}, \mathbf{q})$ is altered by adding a perturbation H_1 with no extra constants of motion, then the Hamiltonian $H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p}, \mathbf{q}) + \varepsilon \cdot H_1(\mathbf{p}, \mathbf{q})$ should have no constants of motion except for the total energy as soon as $\varepsilon \neq 0$. For small ε we should expect well-behaved trajectories to eventually wander off and fill the phase space volume allowed by energy conservation.

The stability of the solar system seems to support the first view while the success of statistical mechanics where the ergodic hypothesis is assumed (trajectories fill up phase space volume without discrimination apart from energy conservation), seems to support the second view, at least for cases when $n \gg 1$. The KAM-theorem shows that the truth lies somewhere between the two extremes.

$$H = H_0 + \varepsilon H_1(p, q) \quad \begin{array}{l} H_0 \text{ Integrable} \\ H_1 \text{ Non-integrable} \end{array}$$

Poincaré showed in 1882 that all constants of motion (except energy) are destroyed when $\varepsilon \neq 0$ but this does not mean that all periodic and quasi-periodic motion of H_0 disappear. Exactly to what degree and how these motions survive is the subject of the KAM-theorem which is center piece of non-linear dynamics. The theorem states that most motion is preserved if ε is small enough.

For $n = 2$ tori whose frequency ratio ω_1/ω_2 are poorly approximated with rational numbers survive when the perturbation is added. The easily approximated ratios are destroyed first.



The total length of destroyed tori goes to zero as $\varepsilon \rightarrow 0$ and the probability that an arbitrary initial condition gives a regular motion goes to one. The first regular tori to be destroyed are those that have ω_1/ω_2 that is close to a rational m/s with s being a small integer. Last to be destroyed are those that have a ratio that is hard to approximate with a rational. Close to being rational can be decided by expansion into continued fractions.

$$\sigma = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \equiv (a_0, a_1, a_2, \dots) \quad \text{Example:}$$

$$\pi = (3, 7, 15, 1, 292, \dots)$$

$$\pi \approx 3 + \frac{1}{7 + \frac{1}{15 + 1/1}} = \frac{355}{113} = 3.1415929$$

Large a_i gives a small relative correction and a long series of small a_i 's mean a good avoidance of being close to a rational number with small denominator. The most irrational number is $\omega^* = (0, 1, 1, 1, \dots) \rightarrow$

$$\omega^* = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \rightarrow \omega^* = \frac{1}{1 + \omega^*} \rightarrow \omega^* = \frac{\sqrt{5} - 1}{2} = 0.61803 \dots \quad \text{The golden mean}$$

Tori with the most irrational frequency ratios are most stable against perturbations. When the golden mean tori are destroyed all trajectories will be chaotic and there is **hard chaos**.

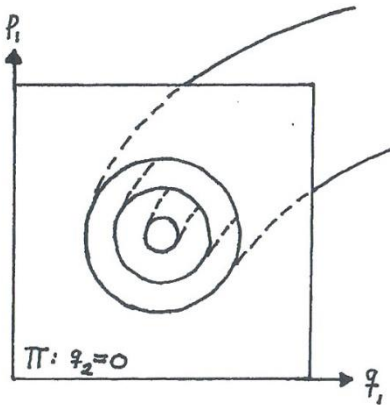
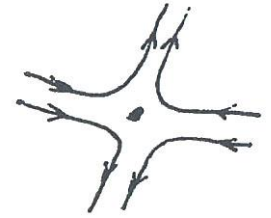
$n = 2$ is a bit special since one torus inside another seal of a region where trajectories are bound to be between the bounding tori even if tori inside are destroyed. Let's look at it and how the KAM-theorem works.

Phase space is filled with tori inside tori, each torus with periodic and quasi-periodic orbits. Look at fixed point in the Poincaré section. The stability properties of the iteration map from one crossing of the Poincaré plane to the next crossing is given by two eigenvalues whose sum is zero (preservation of phase space volume). 2 cases:

1. Two complex conjugate eigenvalues $re^{\pm i\phi}$ → elliptic fixed point.
Stiff pendulum hanging downwards.

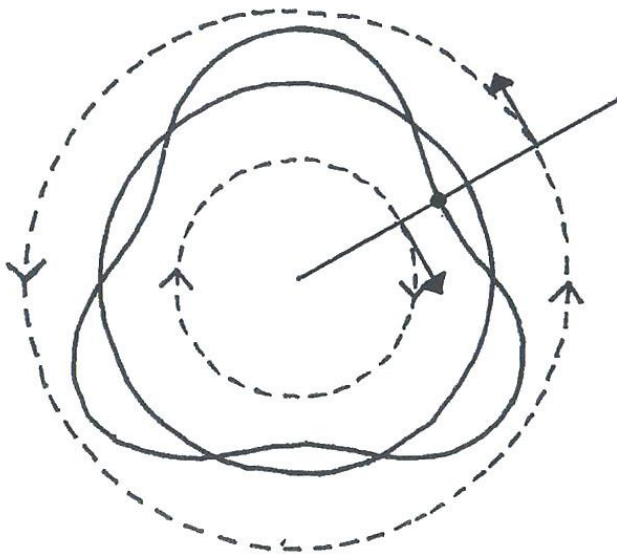


2. One negative and one positive real eigenvalue $\pm r$ → hyperbolic fixed point.
Stiff pendulum hanging upside down.



A 4-dimensional phase space with a 3-dimensional subspace of constant energy and different nested tori inside the subspace with different frequency ratios ω_1/ω_2 .

Consider a torus with rational ω_1/ω_2 . For a periodic orbit with period s the s 'th iterate of the Poincaré map, all points on the circle will be fixed.



If the inner dashed circle has slightly larger ratio ω_1/ω_2 then the outer dashed circle has a slightly larger ratio (or vice versa).

The s 'th iterate of the Poincaré map will move in opposite directions on the outer and inner dashed circle.

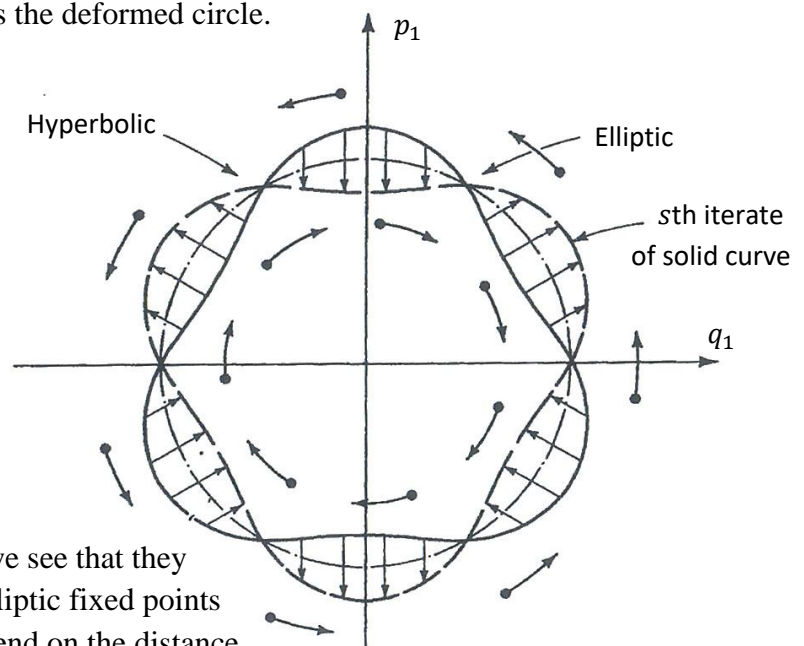
When the perturbation εH_1 is added the rational ω_1/ω_2 -circle no longer maps to itself but points in the inner circle will still move clockwise and outer circle points will map anti-clockwise.

On the ray there will be one point that goes to neither side and stays on the ray. The union of all these points from different rays forms the deformed circle.

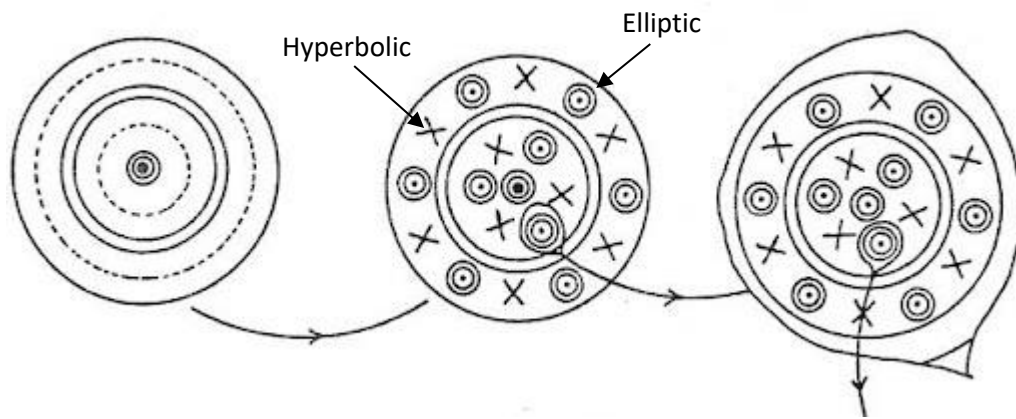
The s 'th iterate of this deformed circle will map to another closed loop encircling the same area (consequence of phase space conservation).

The area preservation between the two curves implies an even number of intersection points. (Poincaré-Birkhoff theorem 1935).

Since the motion of the deformed circle is radial the intersection points will be fixed points.

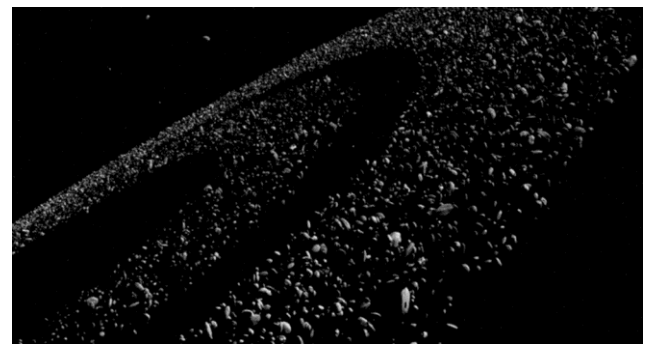
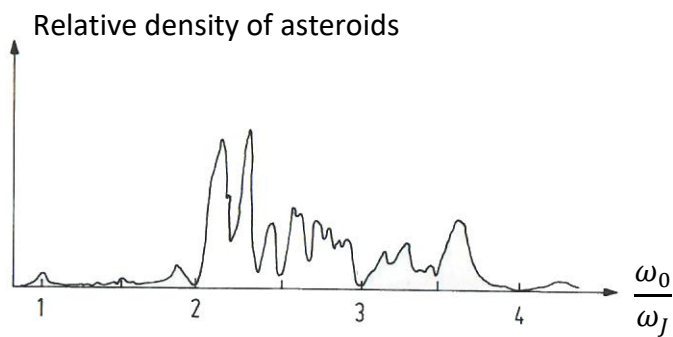


By considering the motion around the fixed points we see that they alternate between elliptic and hyperbolic. Around elliptic fixed points there will be new tori with frequency ratios that depend on the distance to the fixed point. The whole process repeats over and over again in a self-similar way at different scales.



Motion around hyperbolic fixed points give rise to chaotic trajectories while motion around elliptic fixed points is regular. This brings a coexistence of order and chaos with fractal distribution across different scales. As perturbations of an integrable system grow there will be larger parts of phase space filled with chaotic orbits.

An application of the KAM-theorem is the distribution of asteroids in the asteroid belt. The 2-body system of an asteroid's motion around the sun is an integrable system but disturbance from Jupiter makes it a non-integrable 3-body problem. There is the unperturbed frequency of the asteroid motion ω_0 and the frequency of the Jupiter orbit ω_J . A figure of the distribution of frequencies for asteroids in the belt between Mars and Jupiter shows gaps in the distribution when ω_0/ω_J is rational with low denominator just as expected from the KAM-theorem.



Another test of the KAM-theorem is the rings of Saturn with inner satellites as a source of perturbation. Just as for the asteroids we could expect gaps in the frequencies for the orbits of pebbles circling Saturn. Frequency is coupled to orbital radius so we should expect to see gaps in the rings of Saturn where the orbits are likely to be more chaotic and therefore more likely to leave the system. There are gaps, the biggest is the Cassini division.

